C. Wave description leads to the Uncertainty Relation: Important Quantum Sense

- Experiments $\rightarrow$ Wave description of particle $\rightarrow \Psi \rightarrow$ Prob. density $|\Psi|^2$
- Uncertainty relation is a natural by-product

(a) Let's consider $\Psi \sim \sin kx$ (some fixed t)

It looks like ...

- Where is the particle? Can't tell! $\underbrace{|\Psi(x)|^2}_{\text{"}\Delta x\text{" is very large } (\to \infty)}$ spreads over wide range

- $\Psi \sim \sin kx \sim \sin \frac{2\pi x}{\lambda} \sim \sin\left(\frac{px}{\hbar}\right)$

readily known from picture, thus p carries definite value ("Δp" $\to 0$)

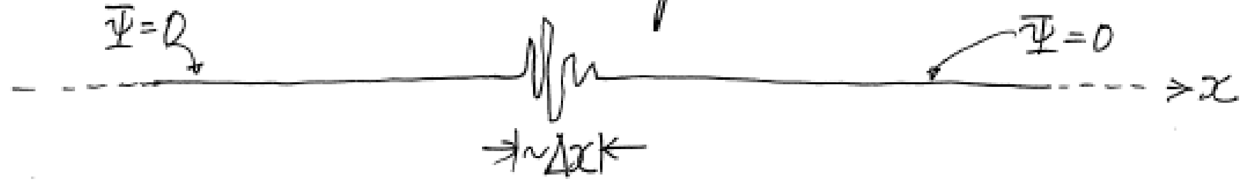
momentum

$\therefore$ When wavelength is known precisely (by showing many repeated patterns),

Δx has to be big

de Broglie $\Rightarrow$ knowing λ = knowing p $\Rightarrow \Delta p \approx 0$

(b) Let's consider $\Psi(x) \sim$ narrow packet



Where is the particle? Easy! $\underbrace{|\Psi|^2 \neq 0 \text{ only in a small range of } x}_{\text{"}\Delta x\text{" is small}}$

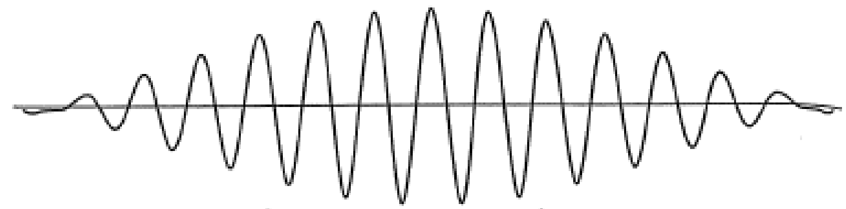
What is the momentum? $\underbrace{\text{Can't tell!}}_{}$

Hard to identify a wavelength

$\Rightarrow$ Hard to identify <u>a</u> momentum

Key concept: a wave packet is formed by adding up waves of different wavelengths

Wave packet

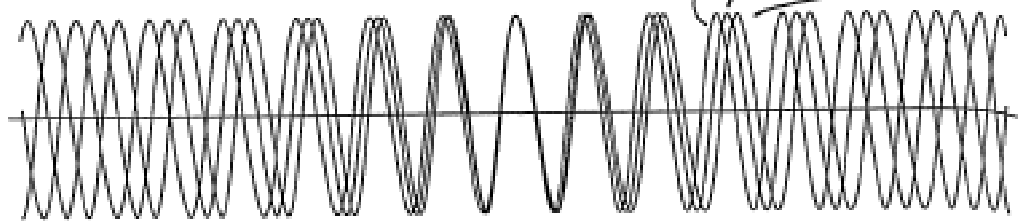


formed by adding

$(\lambda_1 \Rightarrow p_1)\ k_1$

$(\lambda_2 \Rightarrow p_2)\ k_2$

$(\lambda_3 \Rightarrow p_3)\ k_3$



waves of different λ's cancel out

waves of different λ's cancel out

there is a range of wavelengths $\underbrace{\text{(a range of momenta)}}_{\text{thus } \text{"}\Delta p\text{"}}$

Narrower wave packet (Δx smaller) needs more wavelengths (Δp bigger)

$\therefore$ Wave description $\Rightarrow \begin{cases} \text{knowing } p \text{ (thus } \lambda\text{) more precisely, } \Delta x \uparrow \\ \text{knowing location (thus } x\text{) more precisely, } \Delta p \uparrow \end{cases}$

This is the essence of Heisenberg's <u>Uncertainty Relation</u>.

Uncertainty relation is a necessity (unavoidable) once a wave description of matter is adopted

In simple language, it says

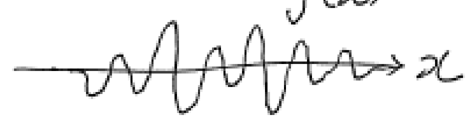
> It is <u>impossible</u> to prepare a particle in a state (thus a $\Psi(x,t)$) in which its momentum and position (along one axis) are exactly known.
> The product of uncertainties[+] Δx and Δp is required to obey
>
> $$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

[+] In QM, there are formal ways to define and calculate Δx and Δp, and (ΔA) is a quantity A for any given state Ψ. We will fill in the Mathematics later.

Look Deeper: The ***Mathematics of Waves*** (or signal processing) is the root

Fourier analysis (the idea)†

- give us a signal



$$\underbrace{f(x)}_{\text{"given signal"}} \underbrace{=}_{\text{"can be decomposed into"}} \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty}}_{\text{"a sum over"}} \underbrace{F(k)}_{\text{"by choosing the weights of different components appropriately"}} \underbrace{e^{ikx}}_{\text{"plane waves of various wavelengths"}} dk$$

"Inverse Fourier transform"

(k is wavevector $= \frac{2\pi}{\lambda}$)

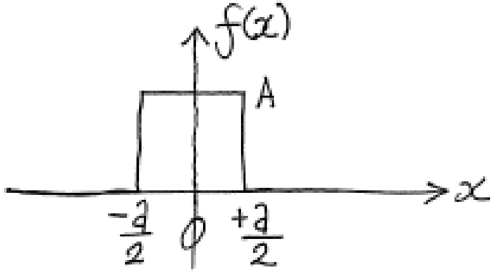
How to choose F(k)?

$$\underbrace{F(k)}_{\text{the right choice}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{f(x)}_{\text{given signal}} e^{-ikx} dx$$

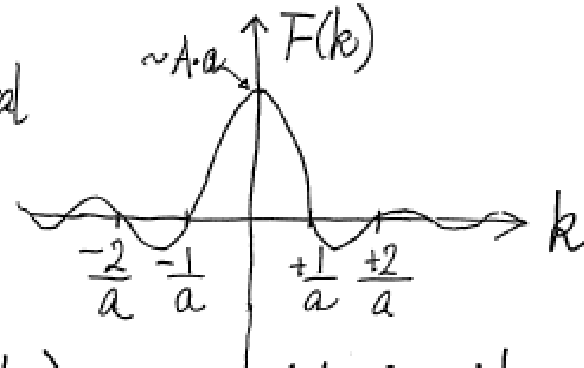
"Fourier transform of f(x)"

† See "The Chemistry Maths Book", Sec. 15.6

A useful example



Do the integral $\longrightarrow$



if square wave is narrower (small a), range of k is wider ($\frac{1}{a}$ is bigger)

So far it is just mathematics of waves

Quantum Physics: $e^{ikx} \sim e^{i\frac{2\pi}{\lambda}x} \sim e^{i\frac{px}{\hbar}}$

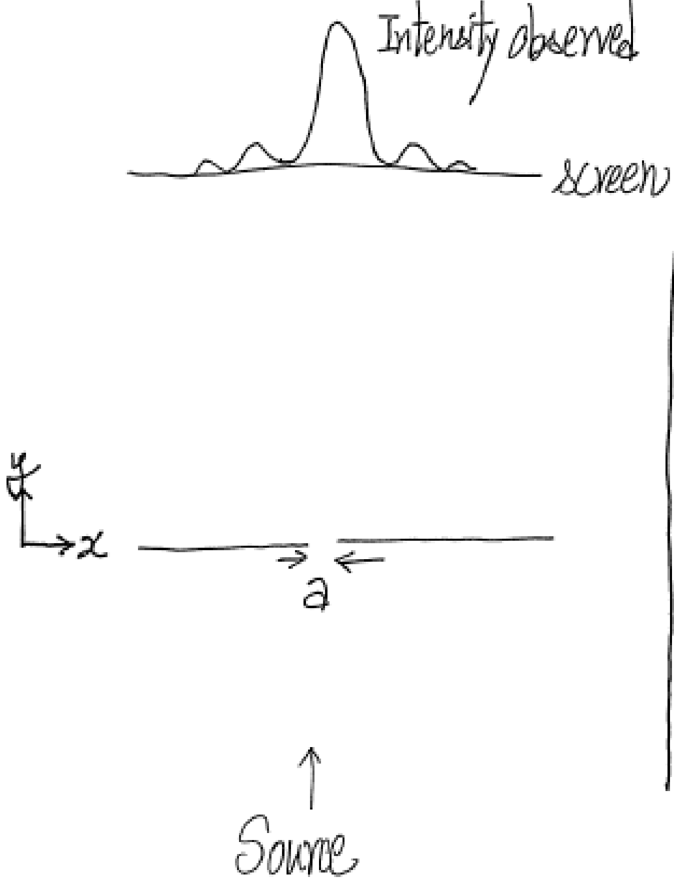
a narrower square wave wavefunction, range of p is wider (Δp bigger)

More: $\Delta x \sim \frac{a}{2}$, $\Delta k \sim \frac{1}{a}$, $\hbar \Delta k \sim \frac{\hbar}{a} \sim \Delta p$

$\therefore \Delta x \cdot \Delta p \sim \frac{\hbar}{2}$ in this example.

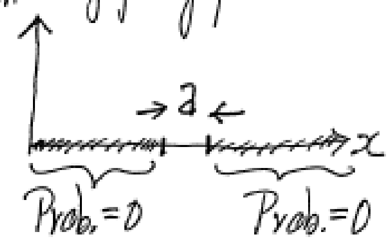
By-Product

- Single-slit Exp't

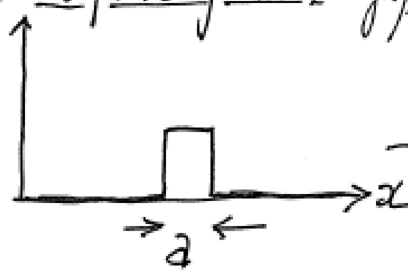


What does the slit do?

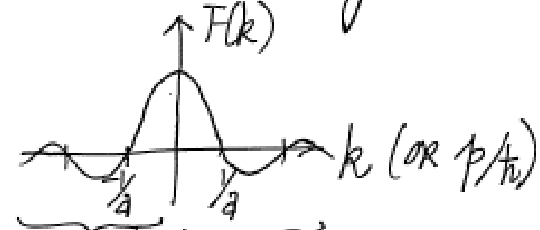
Chance of finding particle



ψ (at plane of slit) [roughly]



- This will propagate from slit to screen (before measuring location)
- How? Fourier analysis



components spreading to $-x$ in time

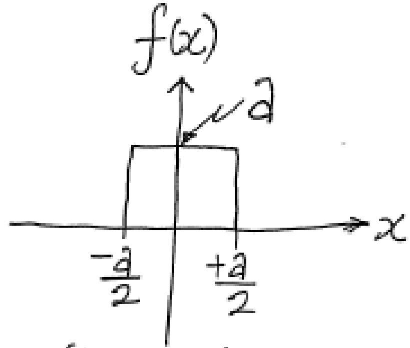
components spreading to $+x$ in time

$\boxed{\text{fan off of } \psi \text{ as } \psi \text{ evolves in time}}$

- Gives intensity pattern for repeated exp'ts
- Can't predict result of one run

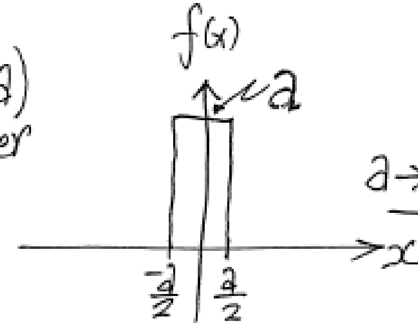
By-Product

- How about ...



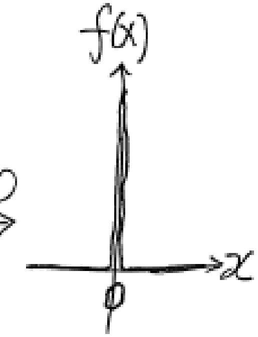
(Area = 1)

(smaller a)
Narrower →



(Area = 1)

$a \to 0$ →



(Area = 1)

$$\begin{pmatrix} f(x) \to \infty, & x = 0 \\ f(x) = 0 & x \neq 0 \end{pmatrix}$$

$$\boxed{\text{Dirac delta function } \delta(x)}$$

Used in QM for a state of definite position (right at $x=0$), e.g. immediately after position measurement

- How many Fourier $\underbrace{\text{k-components}}_{}$ are needed?

(i.e. different momentum components)

Summary

Quantum phenomena need a wave description of matter and thus quantum theory must take in what mathematics of waves has to offer, including the Fourier analysis and thus the Uncertainty Relation

Werner Heisenberg (1901 – 1976)

- PhD (Munich 1923 with Sommerfeld)
- With Born (Gottingen 1 year) and Bohr (3 years) after PhD
- Developed Quantum Mechanics in a form related to Fourier analysis and Matrices (1924-1925) [23 years old – QM was “boys’ physics”]
- Chair of theoretical physics, Univ. of Leipzig (1927 – 1941) (26 years old)
- 1932 Nobel Prize (won the Prize by himself) “for the creation of quantum mechanics” (32 years old)
- Stayed in Germany during WWII
- Participated in Germany’s nuclear bomb project (not sure how hard he tried)
- Re-built German physics after WWII as director of Max Planck Institute for Physics

Joseph Fourier (1768 – 1830)

- Around age 12, discovered mathematics talent
- But wanted to be a priest until 1789
- (Around 1790 – The French Revolution) , almost everybody was involved; Fourier was arrested in 1794 for giving a protest speech
- 1795 – taught at Ecole Polytechnique
- Chair of Analysis and Mechanics (1797), succeeding Lagrange
- Napoleon’s scientific advisor when invading Egypt
- 1822 – Fourier series in his book “Theorie analytique de la chaleur” (Analytic Theory of Heat)

Fourier (among 72 names on Eiffel Tower)